

FINITENESS OF MAPPING DEGREE SETS FOR 3-MANIFOLDS

PIERRE DERBEZ, HONGBIN SUN, AND SHICHENG WANG

ABSTRACT. By constructing certain maps, this note completes the answer of the Question: For which closed orientable 3-manifold N , the set of mapping degrees $\mathcal{D}(M, N)$ is finite for any closed orientable 3-manifold M ?

1. INTRODUCTION

Let M and N be two closed oriented 3-dimensional manifolds. Let $\mathcal{D}(M, N)$ be the set of degrees of maps from M to N , that is

$$\mathcal{D}(M, N) = \{d \in \mathbb{Z} \mid f: M \rightarrow N, \deg(f) = d\}.$$

We will simply use $\mathcal{D}(N)$ to denote $\mathcal{D}(N, N)$, the set of self-mapping degrees of N .

The calculation of $\mathcal{D}(M, N)$ is a classical topic appeared in many literatures. According to [CT], Gromov thought it is a fundamental problem in topology to determine the set $\mathcal{D}(M, N)$ for any dimension n .

The result is simple and well-known for dimension $n = 1, 2$. For dimension $n > 3$, there are some interesting special results (See [DW] for recent ones and references therein), but it is difficult to get general results, since there are no classification results for manifolds of dimension $n > 3$.

The case of dimension 3 becomes the most attractive in this topic. Since Thurston's geometrization conjecture, which has been confirmed, implies that closed orientable 3-manifolds can be classified in reasonable sense.

A basic property of $\mathcal{D}(M, N)$ is reflected in the following:

Question 1. (see also [Re, Problem A] and [W2, Question 1.3]): For which closed orientable 3-manifolds N , the set $\mathcal{D}(M, N)$ is finite for any given closed oriented 3-manifold M ?

The main result proved in this note is the following

Theorem 1.1. *Let N be a given closed oriented 3-manifold N . If $|\mathcal{D}(R)| = \infty$ for each prime factor R of N , then there is a closed orientable 3-manifold M such that $|\mathcal{D}(M, N)| = \infty$.*

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Theorem 1.1 follows from an explicit result Theorem 2.5, which provides the concrete M and the infinite set in $\mathcal{D}(M, N)$ for the given N . The proof of Theorem 1.1 (2.5) is essentially elementary, which does not appear until now mainly due to two reasons:

(1) $|\mathcal{D}(N)|$ may be finite even $|\mathcal{D}(R)| = \infty$ for each prime factor R of N ; for example $|\mathcal{D}(T^3)| = \infty$ but $|\mathcal{D}(T^3 \# T^3)| < \infty$ for 3-dimensional torus T^3 [W1]. Such phenomenon puzzled us to wonder if Theorem 1.1 was always to be true [W2, page 460].

(2) The target concerned in Theorem 1.1 became the only unknown case for Question 1 just very recently. Now Theorem 1.1 completes the answer of Question 1 and we have

Theorem 1.2. *Let N be a closed orientable 3-manifold. Then there is a closed orientable 3-manifold M such that $|\mathcal{D}(M, N)| = \infty$ if and only if $|\mathcal{D}(R)| = \infty$ for each prime factor R of N .*

In the following we will make a brief recall of the development of Theorem 1.2. To be able to do this we need to have a brief look of today's picture of 3-manifolds.

The picture of 3-manifolds: Each closed orientable 3-manifold N has unique prime decomposition $N_1 \# \dots \# N_k$, the prime factors are unique up to the order and up to homeomorphisms. Each closed orientable prime 3-manifold N has a unique geometric decomposition such that each geometric piece supports one of the following eight geometries: H^3 , $\widetilde{PSL}(2, R)$, $H^2 \times E^1$, Sol, Nil, E^3 , S^3 and $S^2 \times E^1$ (where H^n , E^n and S^n are n -dimensional hyperbolic space, Euclidean space and sphere respectively), for details see [Th] and [Sc]. Moreover each geometric piece of N with non-trivial geometric decomposition supports either H^3 -geometry or $H^2 \times E^1$ -geometry, hence each 3-manifold supporting one of the remaining six geometry is closed. Furthermore each 3-manifold supporting geometries of either $H^2 \times E^1$, or E^3 , or $S^2 \times E^1$ is covered by a trivial circle bundle, and each 3-manifold supporting geometries of either Sol, or Nil, or E^3 is covered by a torus bundle. Call prime closed orientable 3-manifold N a *non-trivial graph manifold* if N has non-trivial geometric decomposition but contains no hyperbolic piece.

The development of Theorem 1.2: It is a common sense for many people that $|\mathcal{D}(N)| = \infty$ for 3-manifold N which is either a product of a surface and the circle, or N is covered by the 3-sphere. The first significant result in this direction is due to Milnor and Thurston in the later 1970's. By using the minimum integer number of 3-simplices to build N [MT, Theorem 2], they proved

Theorem 1.3. *For each given hyperbolic 3-manifold N , $|\mathcal{D}(M, N)| < \infty$ for any M .*

Gromov [G] introduced the simplicial volume $\|N\|$ for a manifold N , which is approximately the minimum real number of 3-simplices to build N . Gromov and Thurston proved that $\|N\|$ is proportional to the hyperbolic volume of N in the case of N is a hyperbolic 3-manifold, and then Soma proved $\|N\|$ is proportional to the sum of the hyperbolic volume of the hyperbolic pieces in the geometric decomposition of N (see [G], [Th], [So]). $\| * \|$

respects the mapping degrees, i.e. for any map $f: M \rightarrow N$ then $\|M\| \geq |\deg(f)| \cdot \|N\|$. Then it is deduced that

Theorem 1.4. *Suppose N is a closed orientable 3-manifold. If a prime factor of N having hyperbolic piece in its geometric decomposition, then $|\mathcal{D}(M, N)| < \infty$ for any M .*

Brooks and Goldman [BG1] [BG2] introduced the Seifert volume $SV(*)$ for closed orientable 3-manifolds which also respects the mapping degrees and is non-zero for each 3-manifold supporting the $\widetilde{PSL}(2, R)$ geometry. Then it is deduced that

Theorem 1.5. *Suppose N is a closed orientable 3-manifold. If a prime factor of N supporting $\widetilde{PSL}(2, R)$ geometry. Then $|\mathcal{D}(M, N)| < \infty$ for any M .*

Both Theorems 1.4 and 1.5 were already known in the early 1980's. The following result is known no later than early 1990's (see [W1] for example).

Proposition 1.6. *Suppose N is a closed orientable 3-manifold. Then $|\mathcal{D}(N)| = \infty$ if and only if either N is covered by a torus bundle or a trivial circle bundle, or each prime factor of N is covered by S^3 or $S^2 \times E^1$.*

After Theorems 1.4 1.5 and Proposition 1.6, the remaining unknown cases for Question 1 are: either N is a non-trivial graph manifold; or N is a non-prime 3-manifold, and $|\mathcal{D}(R)| = \infty$ for each prime factor R of N , but some R is not covered by either S^3 or $S^2 \times E^1$.

In 2009 it is proved in [DeW] that each closed orientable non-trivial graph manifold N has a finite covering \tilde{N} with positive Seifert volume (it is still unknown whether $SV(\tilde{N}) > 0$ implies $SV(N) > 0$ for a finite cover $\tilde{N} \rightarrow N$), and therefore it is deduced that

Theorem 1.7. *Let N be closed orientable non-trivial graph manifold. Then $|\mathcal{D}(M, N)| < \infty$ for any closed orientable 3-manifold M .*

Theorems 1.4 1.5, 1.7 and 1.1 (and Proposition 1.6) imply Theorem 1.2.

Remark 1.8. Recently $\mathcal{D}(N)$ is completely determined for each N with $|\mathcal{D}(N)| = \infty$ ([Du], [SWW], [SWWZ]), which is useful in the proof of Theorem 1.1 (2.5).

2. PROOF OF THEOREM 1.1

Call a map $f: M \rightarrow N$ between connected manifolds is π_1 -surjective if the induced $f_*: \pi_1 M \rightarrow \pi_1 N$ is surjective. We start with the following classical fact in topology, whose proof is inspired by Stallings's elegant proof of Grushko's theorem [St] and appeared in several papers (for an easy and recent one, see [RW]).

Lemma 2.1. *Let $f: M \rightarrow N$ be a π_1 -surjective nonzero degree map between closed oriented n -manifolds, with $n \geq 3$. Then for any n -ball B in N , there exists a map g homotopic to f such that $g^{-1}(B)$ is an n -ball in M .*

Denote the subset of $\mathcal{D}(M, N)$ which realized by π_1 -surjective map $f : M \rightarrow N$ as $\mathcal{D}_{surj}(M, N)$. Then the fact below is primary for our construction.

Lemma 2.2. *Suppose $f_i : M_i \rightarrow N_i$ is a π_1 -surjective map of degree d between closed oriented 3-manifolds, $i = 1, \dots, k$. Then there is a π_1 -surjective map $f : M_1 \# \dots \# M_k \rightarrow N_1 \# \dots \# N_k$ of degree d . In particular,*

$$\mathcal{D}_{surj}(M_1 \# \dots \# M_k, N_1 \# \dots \# N_k) \supset \mathcal{D}_{surj}(M_1, N_1) \cap \dots \cap \mathcal{D}_{surj}(M_k, N_k).$$

Proof. Suppose first $k = 2$. Since f_* is π_1 -surjective, by Lemma 2.1, we can homotopy f_i such that for some n -ball $D'_i \subset N_i$, $f_i^{-1}(D'_i)$ is an n -ball $D_i \subset M_i$. Thus we get a proper map $\bar{f}_i : M_i \setminus D_i \rightarrow N_i \setminus D'_i$ of degree d , which also induces a degree d map from ∂D_i to $\partial D'_i$. Since maps of the same degree between $(n-1)$ -spheres are homotopic, so after proper homotopy, we can paste \bar{f}_1 and \bar{f}_2 along the boundary to get map $f = f_1 \# f_2 : M_1 \# M_2 \rightarrow N_1 \# N_2$ of degree d . Moreover $f_* = f_{1*} * f_{2*} : \pi_1 M_1 * \pi_1 M_2 \rightarrow \pi_1 N_1 * \pi_1 N_2$ is surjective since each $f_{i*} : \pi_i M_i \rightarrow \pi_i N_i$ is surjective. Also clearly

$$\mathcal{D}_{surj}(M_1 \# M_2, N_1 \# N_2) \supset \mathcal{D}_{surj}(M_1, N_1) \cap \mathcal{D}_{surj}(M_2, N_2).$$

Then the proof of the Lemma is finished by induction. \square

Suppose $N = N_1 \# \dots \# N_k$ subjects the condition in Theorem 1.1. To apply Lemma 2.2 to prove Theorem 1.1, for each N_i , we need to find a 3-manifold M_i so that $\cap_{i=1}^k \mathcal{D}_{surj}(M_i, N_i)$ is an infinite set. The next lemma provides a uniform and the simplest way to construct such M_i .

Lemma 2.3. *Let M be a closed oriented manifold. Suppose M has a self-map of degree n , i.e., $n \in \mathcal{D}(M)$. Then there is a π_1 -surjective map $g : M \# M \rightarrow M$ of degree $n+1$, i.e., $n+1 \in \mathcal{D}_{surj}(M \# M, M)$.*

Proof. Suppose $f : M \rightarrow M$ is a map of degree n . Pick two copies M_1 and M_2 of M and we construct the following maps

$$M_1 \# M_2 \xrightarrow{q} M_1 \vee M_2 \xrightarrow{id \vee f} M_1 \vee M_2 \xrightarrow{u} M,$$

where q is the quotient map which pinches the 2-sphere defining the connected sum $M_1 \# M_2$ to the point defining the one point union $M_1 \vee M_2$, the map $id \vee f$ restricted on M_1 is the identity and restricted on M_2 is the map f , and the map u sends both M_1 and M_2 to M by orientation preserving homeomorphisms. Let $g = u \circ (id \vee f) \circ q$. Then it is easy to see that on top dimensional homology, g sends the fundamental class $[M_1 \# M_2]$ to $(n+1)[M]$ therefore g of degree $n+1$. Furthermore on the fundamental group g_* sends the free factor $\pi_1(M_1)$ of $\pi_1(M_1 \# M_2) = \pi_1(M_1) * \pi_1(M_2)$ to $\pi_1(M)$ isomorphically, hence g is π_1 -surjective. \square

According to and suggested by Lemma 2.3, we will try to find the infinite intersection of $\mathcal{D}(N_i \# N_i, N_i)$, and to do this we should first find the infinite intersection of $\mathcal{D}(N_i)$. Lemma 2.4 below is prepared for this purpose.

To state Lemma 2.4, we need to slightly reorganize the prime 3-manifolds R with $|\mathcal{D}(R)| = \infty$. According to Proposition 1.6, such R is covered by either a torus bundle, or a trivial circle bundle, or the 3-sphere S^3 . Call a 3-manifold R a torus semi-bundle if R is obtained by identifying the boundaries of two twisted I -bundle over the Klein bottle. Each torus semi-bundle is doubly covered by a torus bundle. Each 3-manifold R covered by a torus bundle must be a torus bundle or a torus semi-bundle if R supports the geometry of E^3 or Sol. But some 3-manifolds supporting Nil geometry are neither torus bundle nor torus semi-bundle [SWWZ]. Each R supporting $H^2 \times E^1$ -geometry has a unique Seifert fibration with n singular fibers of index a_1, \dots, a_n , and we will set $\alpha(R) = |a_1 \dots a_n|$ if $n > 0$ and $\alpha(R) = 1$ if $n = 0$. Now we divide prime 3-manifolds R with $|\mathcal{D}(R)| = \infty$ into the following five classes

- (1) R supports S^3 geometry.
- (2) R supports $H^2 \times E^1$ geometry.
- (3) R is a torus bundles or torus semi-bundle;
- (4) R is a Nil 3-manifold not in (3);
- (5) $R = S^2 \times S^1$.

Lemma 2.4. *Suppose R is a closed oriented prime 3-manifold such that $|\mathcal{D}(R)| = \infty$. Then $\mathcal{D}(R)$ contains a infinite set of integers as below:*

- (1) $\mathcal{D}(R) \supset \{l|\pi_1(R)| + 1 | l \in \mathbb{Z}\}$ if R is covered by S^3 ;
- (2) $\mathcal{D}(R) \supset \{l\alpha(R) + 1, l \in \mathbb{Z}\}$ if R supports $H^2 \times E^1$ -geometry; (3) $\mathcal{D}(R) \supset \{(2l + 1)^2 | l \in \mathbb{Z}\}$ if R is a torus bundle or a torus semi-bundle;
- (4) $\mathcal{D}(R) \supset \{(l)^4 | l \equiv 1 \pmod{12}, l \in \mathbb{Z}\}$ if R supports Nil-geometry but not in Class (3).
- (5) $\mathcal{D}(R) = \mathbb{Z}$ if $R = S^2 \times S^1$.

Proof. (5) is obviously. (1) and (2) are derived from known elementary constructions, and certainly one can also find (1) in [W1] [Du] and [SWWZ] and (2) in [W1] and [SWWZ].

(3) is derived from Theorem 1.6 and Theorem 1.7 of [SWW], and (4) is derived from Theorem 1.4 of [SWWZ]. \square

We are going to prove Theorem 1.1. Suppose N is a closed oriented 3-manifold and $|\mathcal{D}(R)| = \infty$ for each prime factor R of N . By the discussion before Lemma 2.4, we have

$$N = (\#_{i=1}^a P_i) \# (\#_{j=1}^b Q_j) \# (\#_{k=1}^c U_k) \# (\#_{m=1}^d V_m) \# (\#_{p=1}^e S^2 \times S^1),$$

where P_i, Q_j, U_k and V_m are 3-manifolds of types in (1), (2), (3) and (4) respectively, and a, b, c, d, e are integers ≥ 0 .

Theorem 2.5. *Let*

$$d(N, l) = (12 \prod_{i=1}^a |\pi(P_i)| \prod_{j=1}^b \alpha(Q_j) l + 1)^4, \quad l \in \mathbb{Z}.$$

Then $d(N, l) + 1 \in \mathcal{D}_{surj}(N \# N, N)$ for each $l \in \mathbb{Z}$.

Proof. It is easy to present $d(N, l)$ in the following four forms

$$d(N, l) = C_1|\pi_1(P_i)| + 1 = C_2|\alpha(Q_j)| + 1 = (2C_3 + 1)^2 = (12C_4 + 1)^4$$

for some integers C_1, C_2, C_3, C_4 .

Comparing those four forms with (1), (2), (3), (4) of Lemma 2.4 respectively, we have that $d(N, l) \in D(R)$ for each prime factor R in N .

By Lemma 2.3, we have that $d(N, l) + 1 \in \mathcal{D}_{surj}(R \# R, R)$ for each prime factor R in N and each $l \in \mathbb{Z}$.

Notice that

$$(\#_{i=1}^a P_i \# P_i) \# (\#_{j=1}^b Q_j \# Q_j) \# (\#_{k=1}^c U_k \# U_k) \# (\#_{m=1}^d V_m \# V_m) \# (\#_{p=1}^{2e} S^2 \times S^1) = N \# N.$$

By Lemma 2.2, we have that $d(N, l) + 1 \in \mathcal{D}_{surj}(N \# N, N)$ for each $l \in \mathbb{Z}$.

This finishes the proof of Theorem 2.5. \square

Therefore we finish the proof of Theorem 1.1.

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CENTRE DE MATHÉMATIQUES ET D'INFORMATIQUE, TECHNOPOLE DE CHATEAU-GOMBERT, 39,
RUE FRÉDÉRIC JOLIOT-CURIE - 13453 MARSEILLE CEDEX 13

E-mail address: derbez@cmi.univ-mrs.fr

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON NJ 08544 USA

E-mail address: hongbins@math.princeton.edu

DEPARTMENT OF MATHEMATICS, PEKING UNIVERSITY, BEIJING, CHINA

E-mail address: wangsc@math.pku.edu.cn